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Coherent states for transparent potentials

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Abstract. The Darboux transformation operator method is applied to the investigation of coherent states of transparent multisoliton potentials. An isometric correspondence between Hilbert spaces of the states of a free particle and a particle moving in the soliton potential is established. It is shown that the Darboux transformation operator being unbounded and closed cannot realize an isometric mapping between Hilbert spaces. A quasispectral representation of transformation operators in terms of continuous basis sets is obtained. Different families of coherent states of the multisoliton potential are introduced. Measures that realize the resolution of the unity in terms of the projectors on the coherent states vectors are calculated. These measures are defined by ordinary smooth functions for the states obtained with the help of bounded transformation operators and by generalized ones otherwise.

1. Introduction

The concept of coherent states (CS) is widely used in different fields of physics and mathematics (see, for example, [1–3]). In particular, it plays an important role in the Berezin quantization scheme [4], in the analysis of the growth of holomorphic functions [5], in a general theory of phase space quasiprobability distributions [6] and in a quantum state engineering [7]. It is necessary to note that at present no unified definition of such states exists in the literature and different authors mean different things when speaking about them. Nevertheless, a careful analysis (see, for example, [8]) shows that almost all definitions have some common points that can be taken as a general definition of coherent states. Following Klauder [8], by coherent states I mean such states that satisfy the following defining properties.

(a) CS are defined by vectors $\psi_z(x,t)$ which belong to a Hilbert space H of the states of a quantum system with the inner product $\langle\cdot|\cdot\rangle$.
(b) The parameter z takes continuous values from a domain $\mathcal{D}$ of an n-dimensional complex space.
(c) There exists a measure $\mu=\mu(z,\bar z)$ (a bar over a symbol indicates complex conjugation) that provides the resolution of the unity in terms of the projectors on the vectors ψ_z

$$\int_{\mathcal{D}} \mathrm{d}\mu\, |\psi_z\rangle\langle\psi_z| = \mathbb{I}. \qquad (1)$$

(d) CS should be temporally stable.

By temporal stability I mean that the vectors $\psi_z(x,t)$ remain coherent at all times (i.e. satisfy properties (a)–(c) for any moment of time). To satisfy this condition I assume that the functions $\psi_z(x,t)$ are solutions of the Schrödinger equation

$$(\mathrm{i}\partial_t - h_0)\psi_z(x,t) = 0$$

where h_0 is the Hamiltonian of a given quantum system which in general can depend on time. Operator h_0 is supposed to be symmetric on a suitable dense domain in H and to have a unique self-adjoint extension to a wider domain. Equation (1) should be understood in a weak sense, then it is equivalent to the relation

$$\int_{\mathcal{D}} \mathrm{d}\mu\, \langle\psi_a|\psi_z\rangle\langle\psi_z|\psi_b\rangle = \langle\psi_a|\psi_b\rangle$$

which should hold for all $\psi_{a,b}$ from a dense domain in H.

It is necessary to note that the coherent states are not uniquely defined by the properties (a)–(d). It will be shown further that there exist different systems of states satisfying these conditions.

Transparent potentials have many remarkable properties. For instance, a quantum particle propagates without reflection when the shape of its potential energy coincides with that of the transparent potentials. Another remarkable property is that each level in the discrete spectrum (if it exists) of such a potential occupies a preassigned position, which is controlled by values of the parameters the potential depends on. The discrete spectrum levels may even be situated in the middle of the continuous spectrum. In the latter case one has *completely transparent potentials* [9]. A subclass of transparent potentials, namely, so-called soliton potentials, finds a significant application in the soliton theory. There is a marvellous vast literature on this subject. Here I cite only a monograph [10]. Because of their remarkable properties transparent potentials would find an application in pseudopotential theories. Recently, they have been used to describe relaxation processes in the Fermi liquid [11]. It is worthwhile to note that the shape-invariant transparent potentials are well known in supersymmetric quantum mechanics [12].

CS for transparent potentials are very far from being explored. This fact may be explained by the lack of any systematic method for their investigation. A clear algebraic structure related to transparent potentials of a general form has not yet been established (to the knowledge of the author) and therefore well known algebraic methods [1] are not suitable in this context. Ladder operators of sufficiently simple form for the discrete spectrum eigenfunctions do not exist for these potentials and therefore one cannot use the approach of [2] for this purpose. An approach based on the uncertainty relation [13] is, in general, not consistent with property (c) mentioned above and therefore should also be rejected.

Recently, it was demonstrated [14, 15] that in some cases the Darboux transformation operators approach is a useful tool for investigation of CS. This method may be applied in each case when two quantum systems are related to one other by a Darboux transformation operator and when the system of CS (in some sense) is known for one of them.

The method of Darboux transformation operators is intimately related to the supersymmetric quantum mechanics (for a review see [12]) where other approaches are known for investigating the CS. For instance, in the case of shape-invariant or self-similar potentials with the known q-algebraic structure [16] one can immediately construct q-coherent states [17]. Other possibilities are based on the use of annihilation operator [18] or ladder operator [19] approaches.

Let us have an exactly solvable Hamiltonian $h_0 = -\partial_x^2 + V_0(x,t)$ for which the CS $\psi_z(x,t)$ are known. Now one wants to obtain the CS for another Hamiltonian $h_1 = -\partial_x^2 + V_1(x,t)$ related to h_0 by the Darboux transformation operator that will be denoted by L. In general, it should be a *non-stationary* Darboux transformation operator defined by the following intertwining relation [20]:

$$L(\mathrm{i}\partial_t - h_0) = (\mathrm{i}\partial_t - h_1)L.$$

If such an operator L is known, then solutions of the transformed Schrödinger equation determined by h_1 can be easily obtained by the action of the operator L on the solutions of the initial Schrödinger equation associated with h_0. It is clear that the functions $\varphi_z(x,t) = L\psi_z(x,t)$ will satisfy all the properties of the CS enumerated above except perhaps for property (c). One of the main goals of this paper is to prove that in the case of soliton potentials, which is a more famous subclass of the transparent potentials, this property does occur. I would like to mention that this approach has been successfully applied to study the CS of anharmonic oscillator Hamiltonians with equidistant and quasiequidistant spectra [14] and the CS of the non-stationary soliton potentials [21] that are related to the soliton solutions of the Kadomtsev–Petviashvili equation. With the help of this approach a classical counterpart of the Darboux transformation has been formulated and it has been shown that at the classical level this transformation leads to a distortion of the phase space [22]. CS of the one-soliton potential have also been investigated and their supercoherent structure has been revealed [15]. In this paper a detailed analysis of CS for the multisoliton potentials is given. I want to stress that the developed approach is also suitable for other transparent potentials related by the Darboux transformation to these potentials for which the CS are known.

This paper is organized as follows. In section 2 I recall basic results on the free-particle CS in a form appropriate for their application in the following sections. In section 3 the Darboux transformation operator intertwining the free-particle Hamiltonian with the one for the multisoliton potential is analysed as an operator acting in the Hilbert space of the states of the free particle. It is shown that it cannot realize a mapping of Hilbert spaces since it is not defined in the whole Hilbert space and cannot be extended to the whole Hilbert space. Isomeric operators expressed in terms of continuous basis sets similar to these previously proposed by Faddeev [23] and analysed by Pursey [24] for the case of purely discrete basis sets are introduced. These operators realize a polar decomposition of the Darboux transformation operators. A quasispectral representation of the Darboux transformation operator and its inverse in terms of continuous basis sets is obtained. In section 4, different systems of CS for the multisoliton potentials are introduced. It is established that the resolution of the unity exists in every case. Explicit expressions for the measures that realize this equality are found. In the conclusion some possible applications of the obtained results are outlined.

2. Free-particle coherent states

In this section I give a brief overview of the well known properties of the Hilbert space of the states of the free particle (see [25] and references therein) and the corresponding CS [2] we need for subsequent analysis.

Annihilation a and creation a^+ operators

$$a = (\mathrm{i} - t)\partial_x + \mathrm{i}x/2 \qquad a^+ = (\mathrm{i} + t)\partial_x - \mathrm{i}x/2$$

form the Heisenberg–Weil subalgebra of the six-dimensional Schrödinger algebra which is a symmetry algebra of the Schrödinger equation with the zero potential. Solutions of the free-particle Schrödinger equation which are square integrable over the full real axis $\mathbb{R} = (-\infty, +\infty)$ with respect to the Lebesgue measure are the eigenstates of the symmetry operator $K_0 = aa^+ + a^+a$, $K_0\psi_n(x,t) = (2n+1)\psi_n(x,t)$. The explicit expression for these functions is as follows:

$$\psi_n(x,t) = (-\mathrm{i})^n (n! 2^n \sqrt{2\pi})^{-1/2} (1+\mathrm{i}t)^{-1/2} \exp[-\mathrm{i}n \arctan t + y^2(\mathrm{i}t-1)/2] H_n(y)$$
$$y = \frac{x}{\sqrt{2+2t^2}}.$$

Operators a and a^+ are the ladder operators for the basis functions ψ_n: $a\psi_n = \sqrt{n}\psi_{n-1}$, $a^+\psi_n = \sqrt{n+1}\psi_{n+1}$ and $a\psi_0 = 0$.

Let us denote by $\mathcal{L}_0$ the lineal of the functions ψ_n, $n = 0, 1, \ldots$ which is the space of all finite linear combinations of the functions ψ_n with complex coefficients. The operators a and a^+, being linear, are defined for all elements from $\mathcal{L}_0$ and $\mathcal{L}_0$ is invariant with respect to the action of these operators. Since the momentum operator $p_x = -\mathrm{i}\partial_x$ and the initial Hamiltonian h_0 are expressed in terms of a and a^+: $p_x = -(a+a^+)/2$, $h_0 = p_x^2$, these operators are well defined in $\mathcal{L}_0$ and map this space into itself.

The Hilbert space of the states of the free particle, H, is defined as a closure of the lineal $\mathcal{L}_0$ with respect to the measure generated by the inner product $\langle\psi_a|\psi_b\rangle$, $\psi_{a,b} \in \mathcal{L}_0$, which as usual is defined with the help of the Lebesgue integral. The functions ψ_n form an orthonormal basis in H, $\langle\psi_n|\psi_k\rangle = \delta_{nk}$. It is well known [26, 27] that the operators p_x and h_0 initially defined on $\mathcal{L}_0$ have unique self-adjoint extensions and, consequently, are essentially self-adjoint in H. The spectrum of h_0 and p_x is purely continuous. They have common eigenfunctions $\psi_p = \psi_p(x,t)$: $p_x\psi_p = p\psi_p$, $h_0\psi_p = p^2\psi_p$, $p \in \mathbb{R}$, which do not belong to H but belong to a wider space H_- of the linear functionals over H_+, $H_+ \subset H \subset H_-$ (the so-called Gel'fand triplet). One can choose the Hilbert–Schmidt equipment of the space H by letting $H_+ = K_0^{-1}H$, since K_0^{-1} is a Hilbert–Schmidt operator. We refer the reader to [28–30] for more details on rigged Hilbert spaces. The explicit expression for the functions $\psi_p(x,t)$ is well known: $\psi_p(x,t) = (2\pi)^{-1/2}\exp(-\mathrm{i}px - \mathrm{i}p^2 t)$.

The functions ψ_p form an orthonormal and complete (in the sense of generalized functions) basis in H, $\langle\psi_p|\psi_q\rangle = \delta(p-q)$. The completeness condition is expressed symbolically in terms of the projectors onto these functions

$$\int \mathrm{d}p\, |\psi_p\rangle\langle\psi_p| = \mathbb{I}. \tag{2}$$

I do not indicate the limits of integration along the whole real axis. This equality should be understood in a weak sense. This means that it is equivalent to

$$\int \mathrm{d}p\, \langle\psi_j|\psi_p\rangle\langle\psi_p|\psi_k\rangle = \delta_{jk} \qquad j,k = 0,1,\ldots$$

where ψ_k, $k = 0, 1, \ldots$ are orthonormal basis functions in the space H.

The free-particle CS may be obtained by applying a displacement operator in the Heisenberg–Weil group to the vacuum vector ψ_0:

$$\psi_z(x,t) = \exp(za^+ - \bar{z}a)\psi_0(x,t) \qquad z \in \mathbb{C}.$$

These vectors are also the eigenvectors of the annihilation operator $a\psi_z = z\psi_z$. The vectors $\psi_z \in H$ belong to a wider set than $\mathcal{L}_0$. Their Fourier decomposition in terms of the basis ψ_n has the form

$$\begin{aligned} &\psi_z = \Phi \sum\nolimits_n a_n z^n \psi_n \\ &\Phi = \Phi(z,\bar{z}) = \exp(-z\bar{z}/2) \\ &a_n = (n!)^{-1/2} \qquad z \in \mathbb{C}. \end{aligned} \tag{3}$$

The vectors $\psi_z(x,t)$ satisfy all the properties enumerated in the introduction. In particular, the measure $\mathrm{d}\mu = \mathrm{d}\mu(z,\bar{z})$ used in relation (1) is well known: $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y/\pi$, $z = x + \mathrm{i}y$ and the domain of integration $\mathcal{D}$ is the whole complex plane $\mathbb{C}$. In what follows I will not indicate the domain of integration in the integrals over the measures. The integration will always be extended over the whole complex plane. To conclude this section I write down the explicit expression for the free-particle CS

$$\psi_z(x,t) = (2\pi)^{-1/4}(1+\mathrm{i}t)^{-1/2}\exp\left[-\tfrac{1}{4}(z+\bar{z})^2 + \frac{(x+2\mathrm{i}z)^2(\mathrm{i}t-1)}{4(1+t^2)}\right].$$

I use the notation x as the spatial coordinate and as the real part of a complex number z. I hope that it will not cause confusion since these quantities will never appear in the same formula.

3. Darboux transformations and isometric operators

In this section the Darboux transformation operator L is analysed as an operator defined in the Hilbert space H. I would like to stress that this operator is unbounded and cannot be defined over the whole space H. It has a domain of definition which is a subset of H. Moreover, its domain of values does not coincide with H. Therefore, this operator cannot realize mapping between Hilbert spaces contrary to the published assertion [31].

To obtain an N-soliton potential we use the Darboux transformation operator approach elaborated in detail in [10]. The action of this operator on a sufficiently smooth function is defined by the formula

$$L\psi = W^{-1}(u_1,\ldots,u_N)W(u_1,\ldots,u_N,\psi)$$

where W stands for the usual symbol of a Wronskian. In our case the initial potential V_0 does not depend on time. Therefore the functions $u_k = u_k(x,t)$ which are solutions of the initial Schrödinger equation, may be chosen as eigenfunctions of the initial Hamiltonian, $h_0u_k = \alpha_k u_k$, and in general, are not supposed to satisfy any boundary conditions. In this case the transformation operator L does not depend on time and transforms solutions of the initial Schrödinger equation into solutions of the Schrödinger equation with the new time-independent potential

$$V_1 = V_0 - 2\partial_x^2 \log W(u_1,\ldots,u_N).$$

In this paper we need not use the time-dependent Darboux transformation which was proposed by Matveev and Salle (see [10]) and advanced by Bagrov and Samsonov [32].

To obtain the N-soliton potential one should take $V_0 = 0$ and specify the transformation functions u_k as follows [10]:

$$\begin{aligned}
&u_{2k-1} = \cosh(a_{2k-1}x + b_{2k-1})\\
&u_{2k} = \sinh(a_{2k}x + b_{2k})\\
&h_0u_k = -a_k^2u_k \qquad k = 1,2,\ldots,N\\
&a_1 < a_2 < \cdots < a_N.
\end{aligned}$$

The time-dependent phase factors are omitted from these functions since they do not affect all the results. In general, the Wronsky determinant contains $N!$ summands. I would like to stress that in the special case of the soliton potentials this determinant may be substantially simplified and presented as a sum of 2^{N-1} hyperbolic cosines [33]

$$W(u_1,\dots,u_N)=2^{1-N}\sum_{(\varepsilon_1,\dots,\varepsilon_N)}^{2^{N-1}}\varepsilon_2\varepsilon_4\dots\varepsilon_p\prod_{j>i}^{N}(\varepsilon_j a_j-\varepsilon_i a_i)\cosh\left[\sum_{l=1}^{N}\varepsilon_l(a_l x+b_l)\right]$$

where $\varepsilon_i=\pm1$, the value of the subscript p at ε_p should be taken equal to N for even N and to $N-1$ for odd N, the summation runs over all ordered and non-identical sets $(\varepsilon_1,\dots,\varepsilon_N)$ (the sets $(\varepsilon_1,\dots,\varepsilon_N)$ and $(-\varepsilon_1,\dots,-\varepsilon_N)$ are declared to be identical).

It can be shown [10] that the potential thus obtained is regular and bounded from below. This implies that the Hamiltonian $h_1=-\partial_x^2+V_1$ is essentially self-adjoint in H. It has a mixed spectrum. The position of the discrete spectrum levels is defined by the values of the parameters a_k: $E_k=-a_k^2$, and corresponding eigenfunctions have the form [34]

$$\varphi_k=N_kW^{(k)}(u_1,\dots,u_N)/W(u_1,\dots,u_N)$$

$$N_k=\left(\tfrac{1}{2}a_k\prod_{j=1(j\neq k)}^{N}|a_k^2-a_j^2|\right)^{1/2}$$

$$h_1\varphi_k=-a_k^2\varphi_k\qquad k=1,\dots,N$$

where $W^{(k)}(u_1,\dots,u_N)$ is the Wronskian of the functions $u_1,\dots,u_N$ except for the function u_k and the factor N_k is introduced to ensure the normalization of the functions φ_k, $\langle\varphi_k|\varphi_j\rangle=\delta_{kj}$, $k,j=1,\dots,N$. The continuous spectrum corresponds to the semiaxis $E>0$. Continuous spectrum eigenfunctions, $\varphi_p=\varphi_p(x,t)$, $p\in\mathbb{R}$ of the Hamiltonian h_1 may be obtained with the aid of the operator L: $\varphi_p=N_p^{-1}L\psi_p$, where the factor $N_p^{-1}>0$ defined by the relation $N_p^2=(p^2+a_1^2)\dots(p^2+a_N^2)$ is introduced to ensure the normalization of the functions φ_p: $\langle\varphi_p|\varphi_q\rangle=\delta(p-q)$, $h_1\varphi_p=p^2\varphi_p$. The set $\{\varphi_j,\ j=1,\dots,N;\ \varphi_p,\ p\in\mathbb{R}\}$ is complete in H.

Since the operator L is linear, the relation $L\psi_p=N_p\varphi_p$ defines the action of this operator on every ψ of the form

$$\psi(x,t)=\int C(p)\psi_p(x,t)\,\mathrm{d}p \tag{4}$$

where $C(p)$ is a finite continuous function over $\mathbb{R}$. The set of functions of the form (4) is a linear space that I shall denote by $\mathcal{L}_{0p}$ and it is dense in H. (More precisely, it is dense in H_- since these are functionals.) Hence, the action of the operator L is defined for each element from $\mathcal{L}_{0p}$. The image of the space $\mathcal{L}_{0p}$, that I shall denote by $\mathcal{L}_{1p}$ consists of the functions

$$\varphi(x,t)=\int C(p)N_p\varphi_p(x,t)\,\mathrm{d}p.$$

The Darboux transformation operator L together with its Laplace adjoint L^+ has remarkable factorization properties [34, 35],

$$g_0=L^+L=(h_0+a_1^2)\dots(h_0+a_N^2) \tag{5}$$

$$g_1=LL^+=(h_1+a_1^2)\dots(h_1+a_N^2). \tag{6}$$

The functions ψ_p are eigenfunctions of g_0, $g_0\psi_p=N_p^2\psi_p$. This implies that the functions φ_p are eigenfunctions of the operator g_1, $g_1\varphi_p=N_p^2\varphi_p$. The discrete spectrum eigenfunctions of the operator h_1, φ_k, $k=1,\dots,N$ belong to the kernel of the operator g_1, $g_1\varphi_k=0$, $k=1,\dots,N$. This means that the operator g_1 is non-negative in H. Therefore, consider the orthogonal decomposition of the space H: $H = H_0 \oplus H_1$ where H_0 is an N-dimensional space with the basis φ_k, $k = 1, \ldots, N$. The functions φ_p, $p \in \mathbb{R}$ form a basis (in the sense of generalized functions) in H_1. In what follows I shall not consider the space H_0 and restrict my consideration to the space H_1 only. As the operators h_1 and g_1 are restricted to this space, they have a pure continuous spectrum and the operator g_1 is strictly positive. I use the same notations for these operators as operators acting in H_1. Taking into account the spectral decomposition of these operators

$$h_1 = \int \mathrm{d}p\, p^2 |\varphi_p\rangle\langle\varphi_p|$$

$$g_1 = \int \mathrm{d}p\, N_p^2 |\varphi_p\rangle\langle\varphi_p|$$

one can specify their domains of definition. For the operator h_1 it consists of all $\varphi \in H_1$ for which the integral

$$\|h_1\varphi\|^2 = \int \mathrm{d}p\, p^4 |\langle\varphi|\varphi_p\rangle|^2$$

converges, while for the operator g_1 one should demand the convergence of the integral

$$\|g_1\varphi\|^2 = \int \mathrm{d}p\, N_p^4 |\langle\varphi|\varphi_p\rangle|^2.$$

It is clear that the operator g_1 is defined on $\mathcal{L}_{1p}$ and maps this space into itself. Using this fact and the factorization property (6) one can define the action of the operator L^+ on the functions φ_p, $L^+\varphi_p = N_p^{-1} L^+ L \psi_p = N_p \psi_p$, and extend this operator by linearity on the whole space $\mathcal{L}_{1p}$.

It is not difficult to see that the following equality:

$$\langle L\psi_p|\varphi_q\rangle = \langle\psi_p|L^+\varphi_q\rangle$$

holds for all $\psi_p \in \mathcal{L}_{0p}$ and $\varphi_q \in \mathcal{L}_{1p}$. Nevertheless, this does not mean that L^+ is an operator adjoint with respect to the inner product to L, for which the domain of definition is $\mathcal{L}_{0p}$. To find such an operator one has to specify correctly its domain of definition. I shall not look for this domain. Instead I shall give a closed extension $\bar{L}$ of the operator L and then find its adjoint $\bar{L}^+$.

Once we know the bases ψ_p and φ_p in H and H_1, respectively, we can consider isometric operators

$$U = \int \mathrm{d}p\, |\varphi_p\rangle\langle\psi_p|$$

$$U^{-1} = U^+ = \int \mathrm{d}p\, |\psi_p\rangle\langle\varphi_p|.$$

Similar operators have been introduced by Faddeev [23] and considered by Pursey [24] for purely discrete bases. These operators are bounded and defined for all elements from H and H_1, respectively.

Consider now the following operators:

$$\bar{L} = \int \mathrm{d}p\, N_p |\varphi_p\rangle\langle\psi_p| \tag{7}$$

$$\bar{L}^+ = \int \mathrm{d}p\, N_p |\psi_p\rangle\langle\varphi_p|. \tag{8}$$

It is not difficult to specify their domains of definition. For this purpose I use the spectral decomposition of the operator g_0 and its square root

$$g_0 = \int \mathrm{d}p\, N_p^2 |\psi_p\rangle\langle\psi_p|$$
$$g_0^{1/2} = \int \mathrm{d}p\, N_p |\psi_p\rangle\langle\psi_p|. \tag{9}$$

It then follows that

$$\|\bar{L}\psi\|^2 = \|g_0^{1/2}\psi\|^2 = \int \mathrm{d}p\, N_p^2 |\langle\psi|\psi_p\rangle|^2.$$

This means that the domain of definition of $\bar{L}$ coincides with that of $g_0^{1/2}$ and consists of all $\psi \in H$ such that the integral in the right-hand side of this equation converges. The domain of definition of $\bar{L}^+$ coincides with that of the operator $g_1^{1/2}$. It is worthwhile to mention that these domains may be described in terms of conditions imposed on functions that are comprised in these domains (see, for example, [36]) since h_0 and h_1 are operators bounded from below and essentially self-adjoint.

It is clear from formulae (7) and (8) that the operator $\bar{L}^+$ is adjoint to $\bar{L}$ with respect to the inner product and the domains of definition of $\bar{L}$ and $\bar{L}^+$ are well specified. Moreover, $\bar{L}^{++} = \bar{L}$. This implies [26, 27] that the operator $\bar{L}$ is closed. Formulae (7) and (8) give a quasispectral representation of the closed operators $\bar{L}$ and $\bar{L}^+$.

It follows from formulae (7) and (8) that $\bar{L}\psi_p = L\psi_p = N_p\varphi_p$ and $\bar{L}^+\varphi_p = L^+\varphi_p = N_p\psi_p$. This means that $\bar{L}$ is a closed extension of the operator L and $\bar{L}^+$ is a similar extension of the operator L^+ when the domains $\mathcal{L}_{0p}$ and $\mathcal{L}_{1p}$ are taken as their initial domains of definition.

From the spectral decomposition of the operators $g_0^{1/2}$ (9) and $g_1^{1/2}$,

$$g_1^{1/2} = \int \mathrm{d}p\, N_p |\varphi_p\rangle\langle\varphi_p|$$

one obtains the following representations for $\bar{L}$ and $\bar{L}^+$:

$$\bar{L} = Ug_0^{1/2} = g_1^{1/2}U \qquad \bar{L}^+ = g_0^{1/2}U^+ = U^+g_1^{1/2}.$$

Such representations are known as the *polar decompositions* or *canonical representations* of closed operators (see, for example, [27, 37]).

The action of the operator U on the basis ψ_n gives an orthonormal basis in H_1: $\zeta_n = U\psi_n$, $\langle\zeta_n|\zeta_k\rangle = \delta_{nk}$. The functions $\varphi_n = g_1^{1/2}\zeta_n = \bar{L}\psi_n = L\psi_n$, hence, form a basis in H_1 equivalent to an orthonormal (so-called Riesz basis, see, for example, [38]). The operator U is non-local and rather complicated. Therefore, there is no way in which simple explicit expressions can be derived for the functions ζ_n. The functions $\varphi_n(x,t) = L\psi_n(x,t)$ are much simpler but they are not orthogonal to each other: $\langle\varphi_n|\varphi_k\rangle = S_{nk}$. I shall denote by S the infinite matrix with entries S_{nk}. The elements of this matrix can easily be expressed in terms of the elements of another matrix $S^0(a)$ with the entries $S^0_{nk}(a) = \langle\psi_n|h_0 + a^2|\psi_k\rangle$. Using the factorization property (5) one can write

$$S_{nk} = \left[S^0(a_1)S^0(a_2)\ldots S^0(a_N)\right]_{nk}.$$

Taking into account that h_0 is expressed in terms of the ladder operators a and a^+ for the basis functions ψ_n, $h_0 = \frac{1}{4}(a + a^+)^2$, one derives the non-zero elements of the matrix $S^0(a)$: $S^0_{nn}(a) = n/2 + \frac{1}{4} + a^2$, $S^0_{nn+2}(a) = \frac{1}{4}\sqrt{(n+1)(n+2)}$. Hence, we see that the number of non-zero elements in each row and column of the matrix S is finite.

Consider now bounded operators

$$M = \int \mathrm{d}p\, N_p^{-1} |\varphi_p\rangle\langle\psi_p|$$

$$M^+ = \int \mathrm{d}p\, N_p^{-1} |\psi_p\rangle\langle\varphi_p|$$

defined in H and H_1, respectively. It is not difficult to see that $M\bar{L}^+$ is the unit operator in H_1 and $M^+\bar{L}$ is the unit operator in H. Using the spectral resolutions of the operators $g_0^{-1/2}$ and $g_1^{-1/2}$,

$$g_0^{-1/2} = \int \mathrm{d}p\, N_p^{-1} |\psi_p\rangle\langle\psi_p|$$

$$g_1^{-1/2} = \int \mathrm{d}p\, N_p^{-1} |\varphi_p\rangle\langle\varphi_p|$$

one derives the polar decompositions of the operators M and M^+:

$$M = Ug_0^{-1/2} = g_1^{-1/2}U$$

$$M^+ = g_0^{-1/2}U^+ = U^+g_1^{-1/2}.$$

It is easily seen that these operators factorize the operators inverse to g_0 and g_1: $M^+M = g_0^{-1}$, $MM^+ = g_1^{-1}$.

The functions $\eta_n = g_1^{-1/2}\zeta_n = M\psi_n$ form another basis in H_1 equivalent to an orthonormal. This basis is biorthogonal to φ_n, $\langle\varphi_n|\eta_k\rangle = \delta_{nk}$. From this relation and from the factorization property for the operator g_0^{-1}, follows the representation for the elements S_{nk}^{-1} of the matrix inverse to S,

$$S_{nk}^{-1} = \langle\eta_n|\eta_k\rangle = \langle\psi_n|g_0^{-1}|\psi_k\rangle$$

$$= \int \mathrm{d}p\, N_p^{-2}\langle\psi_n|\psi_p\rangle\langle\psi_p|\psi_k\rangle.$$

As a final remark of this section I would like to note the following. The space H_1 can be obtained as a closure of the lineal $\mathcal{L}_1$ of all finite linear combinations of the functions $\varphi_n = L\psi_n$ with respect to the norm generated by the inner product which is a restriction of the given inner product in H to the lineal $\mathcal{L}_1$. The set of functions of the form $\varphi = \bar{L}\psi$ when ψ runs through the whole domain of definition of the operator $\bar{L}$ (i.e. the domain $D_{\sqrt{g_0}}$ of definition of the operator $\sqrt{g_0}$) cannot give the whole space H_1. Nevertheless, if one defines a new inner product in $\mathcal{L}_1$, $\langle\varphi_a|\varphi_b\rangle_1 \equiv \langle L\psi_a|L\psi_b\rangle = \langle\psi_a|g_0|\psi_b\rangle$, $\psi_{a,b} \in \mathcal{L}_0$, $\varphi_{a,b} \in \mathcal{L}_1$ then the closure of $\mathcal{L}_1$ with respect to the norm generated by this inner product coincides with the set $\varphi = \bar{L}\psi$, $\psi \in D_{\sqrt{g_0}}$. This space is embedded in H_1.

4. Coherent states of soliton potentials

The operator g_0 is a symmetry operator for the free-particle Schrödinger equation. Therefore, it commutes with the Schrödinger operator $\mathrm{i}\partial_t - h_0$ when applied to the solutions of the Schrödinger equation. It follows that the operator $U = \bar{L}g_0^{-1/2}$ is an intertwining operator for the Schrödinger operators $U(\mathrm{i}\partial_t - h_0) = (\mathrm{i}\partial_t - h_1)U$ and therefore it is a transformation operator. Hence, being applied to a solution of the initial Schrödinger equation (in our case this is the free-particle Schrödinger equation) it gives a solution of the transformed equation (in our case this is the Schrödinger equation with the N-soliton potential). The functions $\zeta_n = U\psi_n$

and $\zeta_z = U\psi_z$ are then solutions of the Schrödinger equation with the N-soliton potential. The Fourier decomposition of the function ζ_z in terms of the basis $\{\zeta_n\}$ has the same form as that of the function ψ_z in terms of $\{\psi_n\}$

$$\zeta_z = \Phi \sum_n a_n \zeta_n .$$

The vectors ζ_z, $z \in \mathbb{C}$ satisfy all the conditions formulated for CS in the introduction because of the isometric nature of the operator U. The resolution of the unity (1) in the space H_1 in terms of the projectors on ζ_z takes place with the same measure $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}y/\pi$, $z = x + \mathrm{i}y$. One of the deficiencies of these coherent states is that a simple explicit expression for the functions $\zeta_z(x,t)$ does not exist. We may correct this deficiency by introducing new coherent states. Let us act the symmetry operator $g_1^{1/2}$ on the vectors ζ_z. As a result one obtains other solutions of the Schrödinger equation with the N-soliton potential

$$\varphi_z = g_1^{1/2}\zeta_z = \bar{L}\psi_z = \Phi \sum\nolimits_n a_n \varphi_n .$$

It is not difficult to see that the value $\langle\psi_z|g_0|\psi_z\rangle$ is finite. This means that ψ_z belongs to the domain of definition of the operator $\bar{L}$ and the above equality has a meaning. Moreover, these functions are sufficiently smooth and one can apply the differential operator L on them directly. Thus, one obtains a coordinate representation of φ_z. For instance, in the case of the one-soliton potential this representation reads

$$\varphi_z(x,t) = -\tfrac{1}{2}(2\pi)^{-1/4}(1+\mathrm{i}t)^{-3/2}[x + 2\mathrm{i}z + 2a(1+\mathrm{i}t)\tanh(ax)] \\ \times \exp\left[-\frac{(x+2\mathrm{i}z)^2}{4+4\mathrm{i}t} - \tfrac{1}{4}(z+\bar{z})^2\right]. \tag{10}$$

The parameter b is not important for the one-soliton potential and we set $b = 0$. We see that these functions are much simpler than ζ_z and may be analysed without difficulty. For example, it is easily seen that [15] $|\varphi_z(x,t)|^2 = |\varphi_z(-x,-t)|^2$. This property reflects the transparent nature of the one-soliton potential.

Another system of states may be obtained with the help of the transformation operator M. Consider the vectors

$$\eta_z = g_1^{-1/2}\zeta_z = M\psi_z = \Phi \sum\nolimits_n a_n \eta_n .$$

The operator M, being inverse to L, has an integral nature. For the case of the one-soliton potential the integration may be carried out analytically [15]. This yields

$$\eta_z(x,t) = -\tfrac{1}{4}\mathrm{i}\sqrt{\pi}(2\pi)^{-1/4}\,\mathrm{sech}(ax)\exp\left[-\tfrac{1}{4}(z+\bar{z})^2 + a^2(1+\mathrm{i}t)\right] \\ \times\left[\exp(2\mathrm{i}az)\,\mathrm{erfc}\left(a\sqrt{1+\mathrm{i}t} + \frac{x/2+\mathrm{i}z}{\sqrt{1+\mathrm{i}t}}\right)\right. \\ \left. - \exp(-2\mathrm{i}az)\,\mathrm{erfc}\left(a\sqrt{1+\mathrm{i}t} - \frac{x/2+\mathrm{i}z}{\sqrt{1+\mathrm{i}t}}\right)\right] \tag{11}$$

where the parameter b is taken to be zero.

It is worthwhile to mention that all the states $\psi_z(x,t)$, $\varphi_z(x,t)$, $\eta_z(x,t)$ and $\zeta_z(x,t)$ cannot represent non-spreading in time wavepackets. Nevertheless, one can interpret them as coherent states since they satisfy all the properties of such states enumerated in the introduction. Now I shall show that for the vectors φ_z and η_z there exist measures $\mu_\varphi = \mu_\varphi(z,\bar{z})$ and $\mu_\eta = \mu_\eta(z,\bar{z})$ that realize the resolution of the unity in H_1 in terms of the projectors on these vectors.

First consider another continuous basis in H_1: $\eta_p = N_p M\psi_p$, $\langle\eta_p|\eta_q\rangle = \delta(p-q)$, $p, q \in \mathbb{R}$. Since $\{\varphi_p\}$ and $\{\eta_p\}$ are bases in H_1, the resolutions of the unity of the type (1) in terms of the vectors η_z and φ_z are equivalent to the equations

$$\int \mathrm{d}\mu_\eta(z,\bar{z})\,\langle\eta_p|\eta_z\rangle\langle\eta_z|\eta_q\rangle = \delta(p-q)$$

$$\int \mathrm{d}\mu_\varphi(z,\bar{z})\,\langle\varphi_p|\varphi_z\rangle\langle\varphi_z|\varphi_q\rangle = \delta(p-q).$$

Taking into account that the functions ψ_p are the eigenfunctions of g_0 and g_0^{-1}, $g_0\psi_p = N_p^2\psi_p$, $g_0^{-1}\psi_p = N_p^{-2}\psi_p$, one arrives at equations for the measures μ_η and μ_φ

$$(N_pN_q)^{-1}\int \mathrm{d}\mu_\eta\,\langle\psi_p|\psi_z\rangle\langle\psi_z|\psi_q\rangle = \delta(p-q) \tag{12}$$

$$N_pN_q\int \mathrm{d}\mu_\varphi\,\langle\psi_p|\psi_z\rangle\langle\psi_z|\psi_q\rangle = \delta(p-q). \tag{13}$$

Note that the integrals involved in these equations are time independent and hence can be calculated at $t = 0$. Therefore, in what follows I let $t = 0$ and look for the measures independent of time.

The momentum representation of the CS ψ_z is well known

$$\langle\psi_p|\psi_z\rangle = (2/\pi)^{1/4}\Phi\psi_p(z)$$

$$\psi_p(z) = \exp(-p^2 + 2zp - z^2/2) \qquad z = x + \mathrm{i}y.$$

Let us look for the measure μ_η in the form $\mathrm{d}\mu_\eta = \omega_\eta(x)\,\mathrm{d}x\,\mathrm{d}y$, $z = x + \mathrm{i}y$. After performing the integration with respect to y in equation (12) one arrives at an equation for $\omega_\eta(x)$,

$$(2\pi)^{1/2}\int \mathrm{d}x\,\omega_\eta(x)\,F_p(x) = N_p^2\exp(2p^2)$$

$$F_p(x) = \exp(4px - 2x^2).$$

The function N_p^2 is known to be a polynomial in p. Then one concludes that $\omega_\eta(x)$ is a polynomial in x whose coefficients are uniquely defined by the coefficients of the polynomial N_p^2. For instance, for the one-soliton potential one can find

$$\omega_\eta(x) = \left(x^2 + a^2 - \tfrac{1}{4}\right)/\pi.$$

This proves that the states η_z may be interpreted as CS.

We note that the states η_z are defined with the help of the bounded operator $g_0^{-1/2}$. This is the reason why the measure μ_η is expressed in terms of ordinary (non-generalized) functions. Another case takes place for the states φ_z which are defined by the semibounded operator $g_1^{1/2}$. I shall now show that the measure μ_φ is expressed in terms of generalized functions.

Let us look for the measure μ_φ in the form $\mathrm{d}\mu_\varphi = \mathrm{d}y\,\mathrm{d}\omega_\varphi(x)$. The integration in equation (13) with respect to y leads to an equation for the measure $\mathrm{d}\omega_\varphi(x)$,

$$(2\pi)^{1/2}\int \mathrm{d}\omega_\varphi(x)\,F_p(x) = N_p^{-2}\exp(2p^2). \tag{14}$$

First we note that $|F_p(x+\mathrm{i}y)| \leqslant \exp(-dx^2 + by^2)$ where $2 \leqslant d \leqslant b$. This means that $F_p(x)$ belongs to a subspace of the space $S_{1/2}^{1/2}$ of entire functions F such that $|F(x+\mathrm{i}y)| \leqslant \exp(-dx^2 + by^2)$, $0 \leqslant d \leqslant b$ [29]. We look for ω_φ as a functional (i.e. a generalized function) over $S_{1/2}^{1/2}$. (We will see that really this is a functional over a subspace $\overset{\circ}{S}{}_{1/2}^{1/2} \subset S_{1/2}^{1/2}$.)

As is known [29], positive-definite functionals (we look for just such a functional) over $S_{1/2}^{1/2}$ are specified by their Fourier transforms. Let $\tilde\omega_\varphi$ be the Fourier transform of the measure $\omega_\varphi(x)$. This means that an integration of a function $F(x) \in S_{1/2}^{1/2}$ with respect to the measure $\omega_\varphi(x)$ should be replaced by the integration of the Fourier transform $\tilde F(t)$ of this function with respect to the measure $\tilde\omega_\varphi$. In particular,

$$\int \mathrm{d}\omega_\varphi(x)\, F_p(x) = \int \mathrm{d}\tilde\omega_\varphi(t)\, \tilde F_p(t) \tag{15}$$

where $\tilde F_p(t)$ is the Fourier image of the function $F_p(x)$ which in our case can easily be found

$$\tilde F_p(t) = \sqrt{\pi/2}\exp(2p^2 + \mathrm{i}pt - t^2/8).$$

As a result equation (14) yields an equation for $\tilde\omega_\varphi(t)$

$$\pi \int \mathrm{d}\tilde\omega_\varphi(t)\exp(-t^2/8 + \mathrm{i}pt) = N_p^{-2}.$$

It is an easy exercise to show that $\tilde\omega_\varphi(t)$ may be expressed in terms of elementary functions. For this purpose we look for $\tilde\omega_\varphi(t)$ in the form $\mathrm{d}\tilde\omega_\varphi(t) = \rho_\varphi(t)\,\mathrm{d}t$ and use the following representation for the function N_p^{-2}:

$$\begin{aligned} N_p^{-2} &= \sum_{k=1}^{N} \frac{A_k}{\tau + a_k^2} \qquad \tau = p^2 \\ A_k &= \left[(\mathrm{d}N_p^2/\mathrm{d}\tau)_{\tau=-a_k^2}\right]^{-1}. \end{aligned} \tag{16}$$

After some algebra one obtains a formula for $\rho_\varphi(t)$

$$\rho_\varphi(t) = (2\pi)^{-1}\sum\nolimits_{k=1}^{N} \frac{A_k}{a_k}\exp(t^2/8 - a_k|t|). \tag{17}$$

Note that for the function $\rho_\varphi(t)$ of the form (17) there exist in $S_{1/2}^{1/2}$ functions $F(p)$ such that the integral in the right-hand side of equation (15) diverges. The convergence condition for this integral imposes a restriction on the decrease of the integrand function $F(x)$ in the left-hand side of equation (15) as $|x| \to \infty$. This function should satisfy an inequality $|F(x)| \geqslant \exp(-2x^2 - Ax)$ where A is a non-negative constant for every function $F(x) \in S_{1/2}^{1/2}$. I denote the set of functions satisfying this condition by $\mathring{S}_{1/2}^{1/2}\ (\subset S_{1/2}^{1/2})$ which obviously is a linear space.

Thus, we have found the measure μ_φ in terms of the generalized function $\omega_\varphi(x)$ over the space $\mathring{S}_{1/2}^{1/2}$, $\mathrm{d}\mu_\varphi = \mathrm{d}y\,\mathrm{d}\omega_\varphi(x)$, $z = x + \mathrm{i}y$ which is defined by its Fourier transform $\tilde\omega_\varphi$. The integrals with respect to this measure should be calculated as follows:

$$\int \mathrm{d}\mu_\varphi\, \langle\varphi_a|\varphi_z\rangle\langle\varphi_z|\varphi_b\rangle \equiv \int \mathrm{d}t\, \tilde\rho_\varphi(t)\, \tilde F_{ab}(t)$$

where $\tilde F_{ab}(t)$ is the Fourier transform of the function

$$F_{ab}(x) = \int \mathrm{d}y\, \langle\varphi_a|\varphi_z\rangle\langle\varphi_z|\varphi_b\rangle \qquad z = x + \mathrm{i}y.$$

To conclude I would like to comment on the calculation of the norms of the functions η_z and φ_z. The square of the norm of η_z may be calculated with the aid of formula (16) for the function N_p^{-2} and the factorization property of the operator g_0^{-1} in terms of the operators M and M^+

$$\langle\eta_z|\eta_z\rangle = \langle\psi_z|g_0^{-1}|\psi_z\rangle = \int \mathrm{d}p\, N_p^{-2}|\langle\psi_z|\psi_p\rangle|^2.$$

After some algebra one obtains

$$\langle\eta_z|\eta_z\rangle = \sum\nolimits_{k=1}^{N} A_k F_k \qquad z = x + \mathrm{i}y$$

$$F_k = \frac{\sqrt{2\pi}}{a_k} \exp[2(a_k^2 - x^2)]\,\mathrm{Re}\big[\exp(4\mathrm{i}a_k x)\,\mathrm{erfc}(a_k\sqrt{2} + \mathrm{i}\sqrt{2}x)\big].$$

Similarly, the square of the norm of the function φ_z coincides with the expectation value of the operator g_0 in the state ψ_z. For instance, for the one-soliton potential one obtains $\langle\varphi_z|\varphi_z\rangle = \langle\psi_z|g_0|\psi_z\rangle = \frac{1}{4} + a^2 + x^2$, $z = x + \mathrm{i}y$.

5. Conclusion

A classical particle which is decayed by a potential well of an arbitrary shape moves without reflection. For a quantum particle, in general, this is not the case. Nevertheless, there exists a wide class of potentials called transparent potentials for which the scattering process of the quantum particle comes about in some sense in a similar way to those of the classical particle, i.e. without reflection. In my opinion this mysterious phenomenon has, up to now, no appropriate explanation. From a practical point of view the answer to this question is rather important. If at the quantum level we were able to force a signal to propagate without reflection we could decrease the output of the emitted signal. All transparent potentials known at present have a remarkable property. They are related to the zero potential (free particle) by Darboux transformations. Up to recent times it was believed that such potentials have a finite number of discrete spectrum levels. Nevertheless, a method based on an infinite chain of Darboux transformations has been proposed recently [39], with the help of which one can create transparent potentials with an infinite number of discrete spectrum levels. To better understand the nature of transparent potentials we should investigate them in detail. Up to now only scattering states have been available for an analysis of the properties of a quantum particle moving in a soliton potential. This paper opens the door to a more detailed analysis since we now have simple exact solutions of the corresponding Schrödinger equation which are square integrable (see formulae (10) and (11)). These results might find applications in signal analysis (see, e.g., [40]) and in quantum optics [3].

As is well known quantum theory gives a more detailed description of nature than the classical theory. Therefore, a one-to-one correspondence between classical and quantum systems does not exist. This, in particular, is expressed in the fact that the quantization procedure is not unique (canonical quantization, geometric quantization, etc). In this respect the following questions are of interest. What are common points (or differences) between two classical systems for which quantum counterparts are related to each other by the Darboux transformation? Whether the distinction between two classical systems for which a quantization gives quantum systems interrelated by the Darboux transformation is essential? In particular, what are the common points between the classical free particle and a particle moving in a transparent potential? The CS approach make it possible to formulate clear steps in the direction of obtaining an answer to these questions. It permits one to construct a classical mechanics counterpart of a given quantum system and to analyse the properties of such a system. This approach has been realized recently for the potential of the form $x^2 + gx^{-2}$

[22]. It was established that at a classical level the Darboux transformation consists of a distortion of the phase space of the classical system. Moreover, this distortion is consistent with the transformation of the Hamilton function in such a way that the equations of motion remain unchanged. Up to now no approach for the analysis of CS of transparent potentials has been elaborated. In this paper I show that the Darboux transformation operator approach may be used to investigate properties of a subclass of transparent potentials, namely, multisoliton potentials. A next step would be the investigation of other types of transparent potentials and the analysis of the classical counterpart of the quantum system that moves in such a potential. We are planning to carry out these investigations in the near future.

The existence of the resolution of the unity (1) for the states (10) and (11) established in this paper makes it possible to construct a holomorphic representation for the Hilbert space of the states of a particle moving in a soliton potential and for operators acting in it. This gives the possibility to use the well elaborated methods of analytic function theory (see, e.g., [41]) to analyse the properties of soliton potentials. In this approach a number of statements are easier to prove and new properties may be established [42]. Another important consequence of the existence of the resolution of the unity is the possibility to construct a so-called 'phase-space formulation of quantum mechanics' (for a review see [43]) for the particle moving in the soliton potential. Such a formulation is a basis for the reconstruction of a quantum state from information obtained by a set of measurements performed on an ensemble of identically prepared systems which is now an actual problem.

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References

[1] Perelomov A M 1986 *Generalised Coherent States and Their Applications* (Berlin: Springer)
[2] Malkin I A and Man'ko V I 1979 *Dynamical Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka)
[3] Klauder J R and Skagerstam B-S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[4] Berezin F A 1986 *The Method of Second Quantization* (Moscow: Nauka)
[5] Vourdas A 1997 *J. Phys. A: Math. Gen.* **30** 4867
[6] Brif C and Mann A 1998 *J. Phys. A: Math. Gen.* **31** L9
[7] Vourdas A 1996 *Phys. Rev.* A **54** 4544
[8] Klauder J R 1996 *J. Phys. A: Math. Gen.* **29** L293
[9] Stahlhofen A A 1995 *Phys. Rev.* A **51** 934
[10] Matveev V and Salle M 1991 *Darboux Transformations and Solitons* (New York: Springer)
[11] Vogel J, Vogel E and Toepffer C 1985 *Ann. Phys., NY* **164** 463
[12] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[13] Nieto M and Simmons L 1978 *Phys. Rev. Lett.* **41** 207
[14] Bagrov V G and Samsonov B F 1996 *Sov. Phys.–JETP* **82** 593
Bagrov V G and Samsonov B F 1996 *J. Phys. A: Math. Gen.* **29** 1011
[15] Samsonov B F 1996 *Phys. Atom. Nucl.* **59** 720
[16] Spiridonov V 1992 *Phys. Rev. Lett.* **69** 398
Skorik S and Spiridonov V 1993 *Lett. Math. Phys.* **28** 59
[17] Zhedanov A S 1993 *J. Math. Phys.* **34** 2631
Fukui T and Aizawa N 1993 *Phys. Lett.* A **180** 308
Spiridonov V 1995 *Lett. Math. Phys.* **35** 179

[18] Bagchi B, Ganguly A, Bhaumik D and Mitra A 1999 *Mod. Phys. Lett.* A **14** 27
[19] Spiridonov V 1995 *Phys. Rev.* A **52** 1909
Gunker G and Roy P 1999 *Phys. Lett.* A **257** 113
[20] Bagrov V G and Samsonov B F 1996 *Phys. Lett.* A **210** 60
[21] Samsonov B F 1998 *Sov. Phys.–JETP* **87** 1046
[22] Samsonov B F 1997 *J. Math. Phys.* **38** 4492
[23] Faddeev L D 1959 *Usp. Mat. Nauk.* **14** 57
[24] Pursey D L 1986 *Phys. Rev.* D **33** 2267
[25] Miller W Jr 1977 *Symmetry and Separation of Variables* (London: Addison-Wesley)
[26] Smirnov V I 1959 *Advanced Course of Mathematics* vol 5 (Moscow: Phys.-Math. Publishing State House)
[27] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics. 1. Functional Analysis* (New York: Academic)
[28] Gel'fand I M and Shilov G E 1957 1958 *The Generalised Functions* vol 1–3 (Moscow: Phys.-Math. Publishing State House)
[29] Gel'fand I M and Vilenkin N Ya 1961 *The Generalised Functions* vol 4 (Moscow: Phys.-Math. Publishing State House)
[30] Berezin F A and Shubin M A 1983 *The Schrödinger Equation* (Moscow: Moscow State University Press)
[31] Montemayor R 1987 *Phys. Rev.* A **36** 1562
[32] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374
[33] Samsonov B F and Shekoyan L A 1998 *Izv. Vuzov Fiz.* **5** 34 (Engl. transl. *Russ. Phys. J.*)
[34] Bagrov V G and Samsonov B F 1995 *Theor. Math. Phys.* **104** 1051
[35] Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett.* A **174** 273
[36] Kostuchenko A G and Sargsyan I S 1979 *Distribution of Eigenvalues* (Moscow: Nauka)
[37] Dunford N and Schwartz J T 1963 *Linear Operators. Part II. Spectral Theory. Self Adjoint Operators in Hilbert Space* (New York: Interscience)
[38] Gohberg I Ts and Krein M G 1965 *Introduction in the Theory of Linear Nonselfadjoint Operators* (Moscow: Nauka)
[39] Shabat A B 1992 *Inverse Problems* **6** 303
Degasperis A and Shabat A 1994 *Theor. Math. Phys.* **100** 230
[40] Daubechies I 1992 *Ten Lectures in Wavelets (CBMS-NSF Regional Conference Series in Applied Mathematics vol 61)* (Philadelphia, PA: SIAM)
[41] Markushevich A I 1967 *Theory of Analytic Functions* vol 1, 2 (Moscow: Nauka)
[42] Perelomov A M 1973 *Funct. Anal. Appl.* **7** 57
[43] Brif C and Mann A 1998 Phase space formulation of quantum mechanics and quantum state reconstruction for physical systems with Lie-group symmetries *Preprint* quant-ph/9809052 (http://xxx.lanl.gov)